Variable Separation of (2+1)-Dimensional General Sasa-Satsuma System Obtained by Extended Tanh Approach

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By means of an extended tanh approach, new types of variable separated solutions u, v, w with two arbitrary lower-dimensional functions of the (2+1)-dimensional general Sasa-Satsuma (GSS) system are derived. Based on the derived variable separation excitation, abundant localized structures such as dromion, peakon and foldon are revealed by selecting appropriate functions p and q. Finally, some elastic and nonelastic interactions among special folded solitary waves are investigated both analytically and graphically. The explicit phase shifts for all the local excitations offered by the common formula are given and applied to these interactions in detail. – PACS numbers: 01.55.+b; 02.30.Jr.

Key words: Variable Separated Solution; Extended Tanh Approach; General Sasa-Satsuma System.

1. Introduction

Due to the wide applications of soliton theory in mathematics, physics, chemistry, biology, communications, astrophysics and geophysics, the study of integrable models has attracted much attention of many mathematicians and physicists. To find some exact explicit soliton solutions for integrable models is one of the most important and significant tasks. There has been a great amount of activities aiming to find methods for the exact solution of nonlinear differential equations. Such include the Bäcklund transformation [1], Darboux transformation [2], Cole-Hopf transformation [3], various tanh methods [4], various Jacobi elliptic function methods [5, 6], multi-linear variable separation approach [7, 8], Painlevé method [9], homogeneous balance method [10], similarity reduction method [11] and so on.

For a given nonlinear evolution equation

$$\Lambda(U, U_t, U_{x_i}, U_{x_i x_j}, \ldots) = 0 \tag{1}$$

with independent variables, $\varsigma = (t, x_1, x_2, \ldots, x_m)$, and a dependent variable, U, we seek its solutions in the form

$$U = \sum_{i=0}^{n} \alpha_i(\varsigma)\phi^i(\omega(\varsigma)), \quad \omega(\varsigma) = \sum_{i=0}^{m} g_i x_i. \tag{2}$$

Using the ansatz (2), one can obtain many explicit and exact travelling wave solutions of nonlinear evolution equation. The main idea of the approach is, that $\phi(\omega(\varsigma))$ is assumed to be a solution of some equation, such as the cubic nonlinear Klein-Gordon equation ($\phi'^2 = \alpha_4\phi^4 + \alpha_2\phi^2 + \alpha_0$), or a solution of the general elliptic equation ($\phi'^2 = \sum_{i=0}^{4} \alpha_i\phi^i$), where $\alpha_i, i \in (1,2,3,4)$ are all arbitrary constants.

In the extended tanh approach [12, 13], $\phi(\omega(\varsigma))$ is assumed to be a solution of equation

$$\phi'^2 = \alpha_0 + \phi^2. \tag{3}$$

In contrast to (2), here $\omega(\varsigma)$ is not a simple linear combination of variables x_i, but assumed to be a function with the variable separated form

$$\omega(\varsigma) = \varsigma_1(x_1,t) + \varsigma_2(x_2,t) + \varsigma_3(x_3,t) + \ldots, \tag{4}$$

where ς_i are arbitrary functions of the indicated variables.

To determine U explicitly, one may take the following steps: First, similar to the usual tanh approach, determine n by balancing the highest-order nonlinear term(s) and the highest-order partial derivative term(s) in the given nonlinear evolution equation. Second, substitute (2) and (3) into the given equation and collect the coefficients of polynomials of ϕ, then eliminate each coefficient to derive a set of partial differential equations of α_i ($i = 0, 1, \ldots, n$) and ω. Third, solve the system of partial differential equations to obtain α_i and ω. Finally, as (3) with $\alpha_0 = 0$ possesses the solution

$$\phi = -\frac{1}{\omega}, \tag{5}$$

substituting α_i, ω and (5) into (2), one can obtain the solution of the equation in concern.

In this paper, with the help of the extended tanh approach we get variable separated solutions for the (2+1)-dimensional general Sasa-Satsuma (GSS) system. Then necessary and sufficient conditions for the selections of the arbitrary functions appearing in the field formula for the completely or noncompletely elastic interaction will be given.

Among others, the solving process here is different from the multi-linear variable separated approach first put forward by Lou [7]. As we know, the multi-linear variable separated approach has been widely discussed in some previous literature. With the help of a Painlevé-Bäcklund transformation and multi-linear variable separated approach, one can find some characteristic types of localized excitations. Certainly, we also can come to the same conclusion by applying the multi-linear variable separated approach.

2. Variable Separated Solution for the GSS System

The (2+1)-dimensional GSS system is

$$w_t + w_{xxx} + 6w_x u + 3wu_x = 0,$$
$$v_t + v_{xxx} + 6v_x u + 3vu_x = 0, \quad u_y = (wv)_x. \tag{6}$$

Realizing the importance of the GSS system in describing the propagation of ultra-short pulses through fibres, several attempts have been made to find soliton solutions of the GSS system by various methods like Hirota's bilinear method, Painlevé analysis, Bäcklund transformation and inverse scattering technique. The authors of [14] analyzed the singularity structure aspects of this system and confirmed its integrability, using the results of the Painlevé analysis. The new class of solutions includes multiple conoidal periodic waves driven by doubly periodic Jacobian elliptic functions and multiple dromions. They have also explored the concept of fission and fusion of dromions in detail.

Along with the extended tanh method, we assume that the system (6) possesses solutions of the form

$$u(x,y,t) = \sum_{i=0}^{l} a_i \phi^i(\omega), \quad v(x,y,t) = \sum_{j=0}^{m} b_j \phi^j(\omega),$$
$$w(x,y,t) = \sum_{k=0}^{n} c_k \phi^k(\omega), \tag{7}$$

where ϕ satisfies

$$\phi'^2 = \phi^2. \tag{8}$$

Here $\omega \equiv \omega(x,y,t)$, $a_i \equiv a_i(x,y,t)$ $(i = 0,1,\ldots,l)$, $b_j \equiv b_j(x,y,t)$ $(j = 0,1,\ldots,m)$ and $c_k \equiv c_k(x,y,t)$ $(k = 0,1,\ldots,n)$ are functions to be determined later. By balancing the highest-order derivative terms with the nonlinear terms in system (6), we obtain $l = 2$, $m = n = 1$. Then we have

$$u(x,y,t) = a_1(x,y,t)\phi(\omega) + a_2(x,y,t)\phi(\omega)^2,$$
$$v(x,y,t) = b_1(x,y,t)\phi(\omega), \tag{9}$$
$$w(x,y,t) = c_1(x,y,t)\phi(\omega).$$

Inserting (8) and (9) into (6), selecting the variable separated ansatz

$$\omega = p(x,t) + q(y,t), \tag{10}$$

and eliminating all the coefficients of polynomials of ϕ, one gets a set of partial differential equations

$$6c_1 p_x(2a_2 + p_x^2) = 0,$$

$$(6a_2 + 6p_x^2)c_{1x} + (6p_x p_{xx} + 3a_{2x} + 9p_x a_1)c_1 + 6c_0 a_2 p_x = 0,$$

$$(6a_1 + 3p_{xx})c_{1x} + (3a_{2x} + 3p_x a_1)c_0 + (p_{xxx} + q_t + 6p_x a_0 + p_t + 3a_{1x})c_1 + 6c_{0x}a_2 + 3c_{1xx}p_x = 0,$$

$$6c_{0x}a_1 + c_{1t} + c_{1xxx} + 3c_0 a_{1x} + 3c_1 a_{0x} + 6c_{1x}a_0 = 0,$$

$$c_{0xxx} + 3c_0 a_{0x} + 6c_{0x}a_0 + c_{0t} = 0,$$

$$6b_1 p_x(2a_2 + p_x^2) = 0,$$

$$(6a_2 + 6p_x^2)b_{1x} + (6p_x p_{xx} + 3a_{2x} + 9p_x a_1)b_1 + 6b_0 a_2 p_x = 0,$$

$$(6a_1 + 3p_{xx})b_{1x} + (3a_{2x} + 3p_x a_1)b_0 + (p_{xxx} + q_t + 6p_x a_0 + p_t + 3a_{1x})b_1 + 3b_{1xx}p_x + 6b_{0x}a_2 = 0,$$

$$6b_{1x}a_0 + 3b_0 a_{1x} + b_{1t} + 6b_{0x}a_1 + 3b_1 a_{0x} + b_{1xxx} = 0,$$

$$3b_0 a_{0x} + b_{0xxx} + b_{0t} + 6b_{0x}a_0 = 0,$$

$$2(a_2 q_y - c_1 p_x b_1) = 0,$$

$$-c_{1x}b_1 - c_1 p_x b_0 + a_1 q_y + a_{2y} - c_0 b_1 p_x - c_1 b_{1x} = 0,$$

$$-c_{0x}b_1 - c_0 b_{1x} + a_{1y} - c_1 b_{0x} - c_{1x}b_0 = 0 a_{0y} - c_{0x}b_0 - c_0 b_{0x} = 0.$$

We obtain the following expressions for the coefficients:

$$\begin{aligned}
&a_2 = -\frac{1}{2}(p_x)^2, \quad b_1 = -\frac{1}{2}\frac{\sqrt{p_x q_y}}{\lambda},\\
&c_1 = \lambda\sqrt{p_x q_y}, \quad a_0 = \frac{3p_{xx}^2 - 4p_x p_{xxx} - 4p_x p_t}{24p_x^2},\\
&c_0 = 0, \quad b_0 = 0, \quad a_1 = -\frac{1}{2}p_{xx}, \quad q = q(y),\\
&\lambda = \lambda(y) \text{ or } \lambda \equiv \text{ constant.}
\end{aligned} \tag{11}$$

Consequently, the exact variable separated solution of the GSS system (6) has the form

$$u = \frac{-3p_{xx}^2 + 4p_x p_{xxx} + 4p_x p_t}{24p_x^2} + \frac{p_{xx}}{2(p+q)} + \frac{p_x^2}{2(p+q)^2}, \tag{12}$$

$$w = -2\lambda^2 v = -\lambda\frac{\sqrt{p_x q_y}}{p+q}, \tag{13}$$

where $p \equiv p(x,t)$ and $q \equiv q(y)$ are arbitrary functions of the indicated variables, $\lambda \equiv \lambda(y)$ is an arbitrary function of y which also may be a constant. Since the fields w and v for the GSS system are real, we have to put a constraint:

$$p_x q_y \geq 0. \tag{14}$$

Obviously, because $p(x,t)$ and $q(y)$ are arbitrary functions, we can set them as

$$p(x,t) = \chi(x) + \tau(t), \quad q(y) = Y(y).$$

Then one of the special conditions is

$$\omega = p(x,t) + q(y) = X(x) + T(t) + Y(y), \tag{15}$$

where $X \equiv X(x)$, $T \equiv T(t)$ and $Y \equiv Y(y)$ are three arbitrary functions of x, t and y, respectively. Under these conditions, the time and space variables are separated entirely. Then we have

$$\begin{aligned}
u &= \frac{-3X_{xx}^2 + 4X_x X_{xxx} + 4X_x T_t}{24X_x^2} + \frac{X_{xx}}{2(X+T+Y)} + \frac{X_x^2}{2(X+T+Y)^2},\\
w &= -2\lambda^2 v = -\lambda\frac{\sqrt{X_x Y_y}}{X+T+Y}.
\end{aligned} \tag{16}$$

It should be mentioned that the variable separated result in (13), i. e.

$$w^2(v^2) \propto \frac{p_x q_y}{(p+q)^2},$$

had also been found for other (2+1)-dimensional models, such as the Boiti-Leon-Pempinelli system, the Korteweg-de Vries equation, the dispersive long wave equation, and the Nizhink-Novikov-Veselov equation.

3. Special Localized Excitations

Because of the arbitrariness of the functions of p and q, (13) reveals quite abundant soliton structures. Apart from the traditional nonpropagating localized excitations like lumps or dromions, many more excitations can be constructed like peakons, compactions, etc.

For example, when selecting in (13) $T = \exp(\cos(t))$ and p and q to be some piecewise smooth functions, we can derive some multi-peakon excitations, i. e.,

$$\begin{aligned}
p &= \exp[\cos(t)] + \sum_{i=1}^{N}\begin{cases} X_i(x), & x \leq 0,\\ -X_i(-x) + 2X_i(0), & x > 0,\end{cases}\\
q &= \sum_{j=1}^{M}\begin{cases} Y_j(y), & y \leq 0,\\ -Y_j(-y) + 2Y_j(0), & y > 0,\end{cases}
\end{aligned} \tag{17}$$

where the functions $X_i(x)$ and $Y_j(y)$ are differentiable functions of the indicated arguments and possess the boundary conditions

$$\begin{aligned}
&X_i(\pm\infty) = A_{\pm i},\ i = 1,2,\ldots,N,\\
&Y_j(\pm\infty) = B_{\pm j},\ j = 1,2,\ldots,M,
\end{aligned} \tag{18}$$

with $A_{\pm i}$ and $B_{\pm j}$ are constants.

Similarly, if p and q are chosen to be other types of piecewise smooth functions, then we can derive multi-compacton excitations, i. e.,

$$\begin{aligned}
p &= \exp[\cos(t)] + \sum_{i=1}^{N}\begin{cases} 0, & x \leq x_{1i},\\ E_i(x) - E_i(x_{1i}), & x_{1i} < x \leq x_{2i},\\ E_i(x_{2i}) - E_i(x_{1i}), & x > x_{2i},\end{cases}\\
q &= \sum_{j=1}^{M}\begin{cases} 0, & y \leq y_{1j},\\ F_j(y) - F_j(y_{1j}), & y_{1j} < y \leq y_{2j},\\ F_j(y_{2j}) - F_j(y_{1j}), & y > y_{2j},\end{cases}
\end{aligned} \tag{19}$$

where the functions E_i and F_j are all arbitrary differentiable functions with the conditions

$$\frac{\partial E_i}{\partial x}|_{x=x_{1i}} = \frac{\partial E_i}{\partial x}|_{x=x_{2i}} = 0, \quad \frac{\partial F_j}{\partial y}|_{y=y_{1j}} = \frac{\partial F_j}{\partial y}|_{y=y_{2j}} = 0. \tag{20}$$

Then the physical fields w and v will become some higher-dimensional multiple compacton solutions. Actually, from (13) it is easy to conclude that for arbitrary p and q with the boundary conditions

$$p|_{x\to-\infty} \to L_1, \quad p|_{x\to+\infty} \to L_2, \quad q|_{y\to-\infty} \to L_3, \quad q|_{y\to+\infty} \to L_4, \tag{21}$$

where L_i $(i = 1,2,3,4)$ are arbitrary constants, which may even be infinities, we obtain a coherent soliton solution localized in some or in all directions. The different choices of the arbitrary functions p and q correspond to the different choices of boundary conditions. On the other hand, one could investigate the stability properties of these solutions and their relevance as asymptotic states for suitable initial boundary value problems.

But we can not obtain the ring type of solitons and lumps from (13) because of the condition $p_x q_y \geq 0$ to guarantee real values of w and v.

4. Asymptotic Properties of the Localized Excitations

In this section, we focus our attention on whether the interactions of these types of localized excitations are completely elastic. To find the answer, we have to study the asymptotic properties of the localized excitations.

If the function q is chosen as an arbitrary static function while p is chosen as a multi-localized solitonic excitation with

$$p|_{t\to\mp\infty} = \sum_{j=1}^{M} h_j^{\mp} \equiv h_j(x - v_j t + \Delta_j^{\mp}), \tag{22}$$

where h_j are localized functions, then the quantities in (12) and (13) are

$$u|_{t\to\mp\infty} \to \frac{3h_{jxx}^{\mp2} - 4h_{jx}^{\mp}h_{jxxx}^{\mp} - 4h_{jx}^{\mp}h_{jt}^{\mp}}{24h_{jx}^{\mp2}} - \frac{h_{jxx}^{\mp}}{2(h_j^{\mp} + H_j^{\mp} + q)} - \frac{h_{jx}^{\mp2}}{2(h_j^{\mp} + H_j^{\mp} + q)^2},$$

$$v|_{t\to\mp\infty} \to \frac{\sqrt{h_{jx}^{\mp} q_y}}{-2\lambda(h_j^{\mp} + H_j^{\mp} + q)},$$

$$w|_{t\to\mp\infty} \to \frac{\lambda\sqrt{h_{jx}^{\mp} q_y}}{h_j^{\mp} + H_j^{\mp} + q}. \tag{23}$$

Here

$$H_j^{\mp} = \sum_{i<j} h_i(\mp\infty) + \sum_{i>j} h_i(\pm\infty), \tag{24}$$

and we have assumed, without loss of generality, $v_j \geq v_i$ if $j \geq i$.

It can be deduced from expression (23) that the localized excitation preserves its shape during the interaction if

$$H_j^{+} = H_j^{-}. \tag{25}$$

The phase shift of the j-th localized excitation in the x-direction reads

$$\Delta_j^{+} = \Delta_j^{-}. \tag{26}$$

The above discussion demonstrates that multiple localized solitonic excitations for the field (13) can be constructed without difficulties via the (1+1)-dimensional multiple localized excitations with the properties (22).

Fortunately, owing to the arbitrariness of the functions in (13), we have constructed not only the single-valued localized excitations but also quite rich folded solitary waves, which are so-called loop solitons in (1+1)-dimensional integrable systems. As a matter of fact, take p_x as (1+1)-dimensional localized multi-valued functions (say loop solitons),

$$p_x \equiv \sum_{j=1}^{M} h_j(\xi - v_j t), \quad x = \xi + \sum_{j=1}^{M} g_j(\xi - v_j t), \tag{27}$$

and the function q in a similar way,

$$q_y = \sum_{j=1}^{M} Q_j(\eta), \quad y = \eta + R(\eta), \tag{28}$$

where $v_1 < v_2 < \ldots < v_M$ are all arbitrary constants and $h_j, g_j, \forall j$ are all localized functions with the properties

$$h_j(\pm\infty) = H^{\pm}, \quad g_j(\pm\infty) = G^{\pm} = \text{ constant}, \tag{29}$$

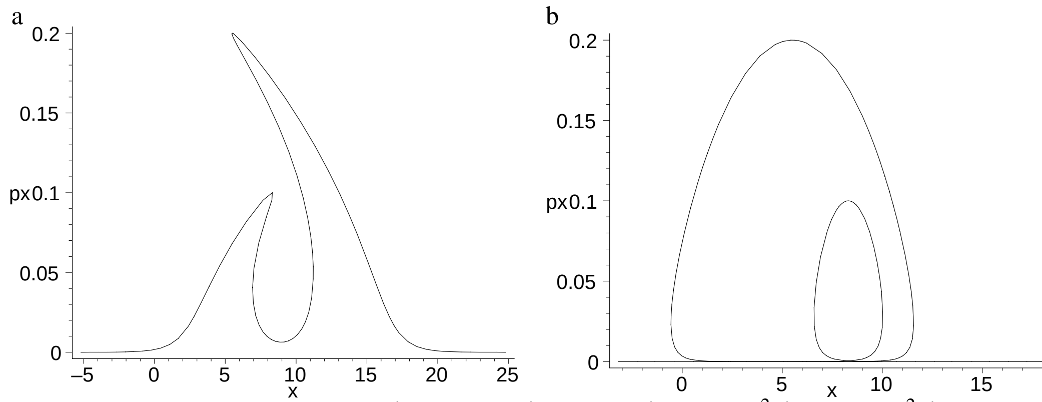

Fig. 1. Plots of p_x with (a) $p_x = 1+0.1\text{sech}(\xi)+0.2\text{sech}(\xi-9)$, $x = \xi - 3.5\tanh^2(\xi)+8.3\tanh^2(\xi-9)$; (b) $p_x = 1+0.1\text{sech}^2(\xi)+0.2\text{sech}^2(\xi-9)$, $x = \xi-3.5\tanh(\xi)+8.3\tanh(\xi-9)$.

then we have

$$p = \int^{\xi} p_x x_\xi \mathrm{d}\xi, \quad q = \int^{\eta} q_y y_\eta \mathrm{d}\eta. \tag{30}$$

Substituting (27) and (28) with (30) into (13), we can get folded solitary waves for $w(v)$. As can be concluded from expression (27), ξ may be a multi-valued function in certain regions of x by selecting the functions g_j suitably. Therefore, the function p_x may be a multi-valued function of x in these regions though it is a single-valued function of ξ, see Figure 1. A more detailed choice of the function is given in the figure caption. Besides, p_x is an travelling solution of M localized excitations due to the property $\xi|_{x\to\infty} \to x \to \infty$. In this case, the phase factors $\Delta_j^{\mp}$ read

$$\Delta_j(\pm) = \sum_{i<j} G_i^{\mp} + \sum_{i>j} G_i^{\pm}. \tag{31}$$

To construct folded solitary waves with completely elastic interaction properties for $w(v)$, we have to discuss its asymptotic properties when $t \to \pm\infty$. Assuming further that $H_j(\theta_j) \equiv H_j(\xi - v_j t) \equiv \int h_j \mathrm{d}x|_{\theta_j\to\pm\infty} \to H_j^{\pm}$, we can look at the j-th excitation to see the interaction properties among the localized excitations expressed by (13), (27) and (28). In other words, we can consider the θ_j as invariant and then take $t\to\infty$ because q has been fixed as t-independent. The results read

$$w_j^2 = 4\lambda^4 v_j^2|_{t\to\mp\infty} \to \lambda^2 \frac{h_j(\theta_j) q_y}{(H_j(\theta_j)+\Omega_j^{\mp}+q)^2}, \quad x|_{t\to\mp\infty} \to \xi + \Delta_j^{\mp} + g_j(\xi + v_j t), \tag{32}$$

where

$$\Omega_j^{\mp} = \sum_{i>j} H_i^{\mp} + \sum_{i<j} H_i^{\pm}, \quad \Delta_j(\mp) = \sum_{i>j} G_i^{\mp} + \sum_{i<j} G_i^{\pm}. \tag{33}$$

Only for a special choice of the spectral parameters these solutions preserve their forms.

5. Examples of the Elastic and Nonelastic Interactions

Now we plot and discuss two concrete interactions for the field w with and without completely elastic interaction properties. In order to reveal the phase shift more clearly and visually, it has proved convenient to fix one of them possessing zero velocity.

5.1. Nonelastic Interaction of Two-Folded Solitary Waves of w

For instance, Fig. 2 displays a pre- and post-interaction plot of the two-folded solitary waves for the field w expressed by (13) with the choices

$$p_x = 0.8\text{sech}(\xi)^2 + 1.6\text{sech}(\xi - 0.5t)^2,$$

$$p = \int^{\xi} p_x x_\xi \mathrm{d}\xi = \{(\alpha+\beta)(\beta-1)^3(\alpha+1)^3\}^{-1}[-1.6\alpha(\beta-1)^3(\alpha-1)(\alpha+\beta) - 3.2\beta(\alpha+1)^2(\beta-1)(\alpha\beta^2+4\alpha\beta+7\alpha+1+\beta^2+10\beta) + 19.2\beta\ln\frac{\alpha+\beta}{\alpha+1}(\beta+1)(\alpha+\beta)(\alpha+1)^3] + 6;$$

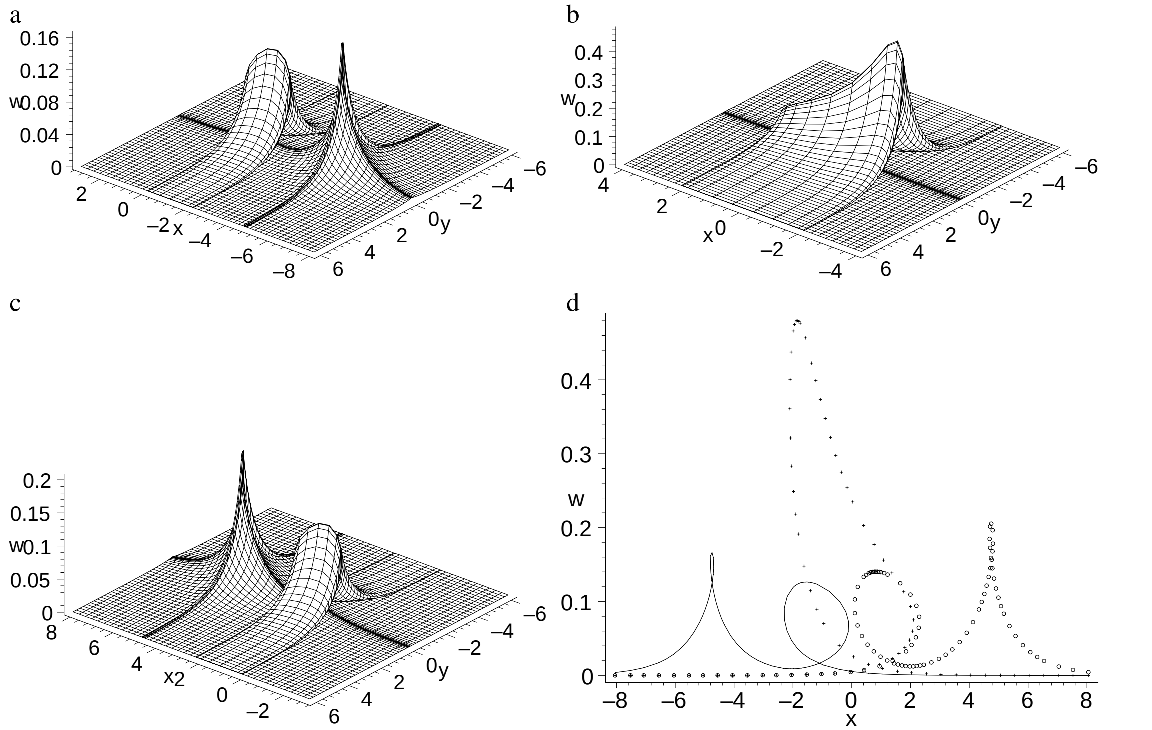

Fig. 2. Nonelastic interaction between special folded solitary waves w with conditions (34) at (a) $t = -15$; (b) $t = 0$; (c) $t = 15$. (d) Corresponding sectional view at $y = 0$; solid line, dots, circles denote before, in and after collision, respectively.

$$\alpha \equiv \exp(2\xi), \quad \beta \equiv \exp(t),$$
$$x = \xi - 2.75\tanh(\xi) - 1.2\tanh(\xi - 0.5t),$$
$$q_y = 0.85\mathrm{sech}(\eta)^2,$$
$$q = \int^{\eta} q_y y_\eta \mathrm{d}\eta = \tanh(\xi)[0.34 - 0.255\mathrm{sech}(\xi)^2],$$
$$y = \eta - 0.9\tanh(\eta). \tag{34}$$

From Figs. 2a–c, we conclude that the field w obtained from (34) expresses special two-folded solitary wave solutions in which the interaction between them is nonelastic. The total phase shift for the static folded solitary wave is

$$\varDelta_1^+ - \varDelta_1^- = G_2(-\infty) - G_2(+\infty) = 2.4. \tag{35}$$

Actually, the condition (25) for completely elastic interaction is not satisfied. For the static folded solitary wave we have

$$\varOmega_1^+ - \varOmega_1^- = H_2(-\infty) - H_2(+\infty) = -\frac{16}{25} \neq 0, \tag{36}$$

and for the moving folded solitary wave we obtain

$$\varOmega_2^+ - \varOmega_2^- = H_1(+\infty) - H_1(-\infty) = -\frac{4}{3} \neq 0. \tag{37}$$

5.2. Elastic Interaction of Two-Folded Solitary Waves of w

The interaction property of two-folded solitary waves is shown in Fig. 3 with the choices

$$p_x = 1.3\mathrm{sech}(\xi)^2 + 0.5\mathrm{sech}(\xi - t)^2,$$
$$\begin{aligned} p = \int^{\xi} p_x x_\xi \mathrm{d}\xi &= \{(\alpha+\beta)(\beta-1)^3(\alpha+1)^3\}^{-1} \\ &\cdot [-2.6\alpha(\beta-1)^3(\alpha-1)(\alpha+\beta) - \beta(\alpha+1)^2(\beta-1) \\ &\cdot (\alpha\beta^2 + 4\alpha\beta + 7\alpha + 1 + \beta^2 + 10\beta) \\ &+ 6\beta\ln\frac{\alpha+\beta}{\alpha+1}(\beta+1)(\alpha+\beta)(\alpha+1)^3] + 8; \end{aligned}$$
$$\alpha \equiv \exp(2\xi), \quad \beta \equiv \exp(2t),$$
$$x = \xi - 1.5\tanh(\xi) - 1.5\tanh(\xi - t),$$
$$q_y = 2.8\mathrm{sech}(\eta)^2,$$
$$q = \int^{\eta} q_y y_\eta \mathrm{d}\eta = -1.4\tanh(\xi)\mathrm{sech}(\xi)^2,$$
$$y = \eta - 1.5\tanh(\eta). \tag{38}$$

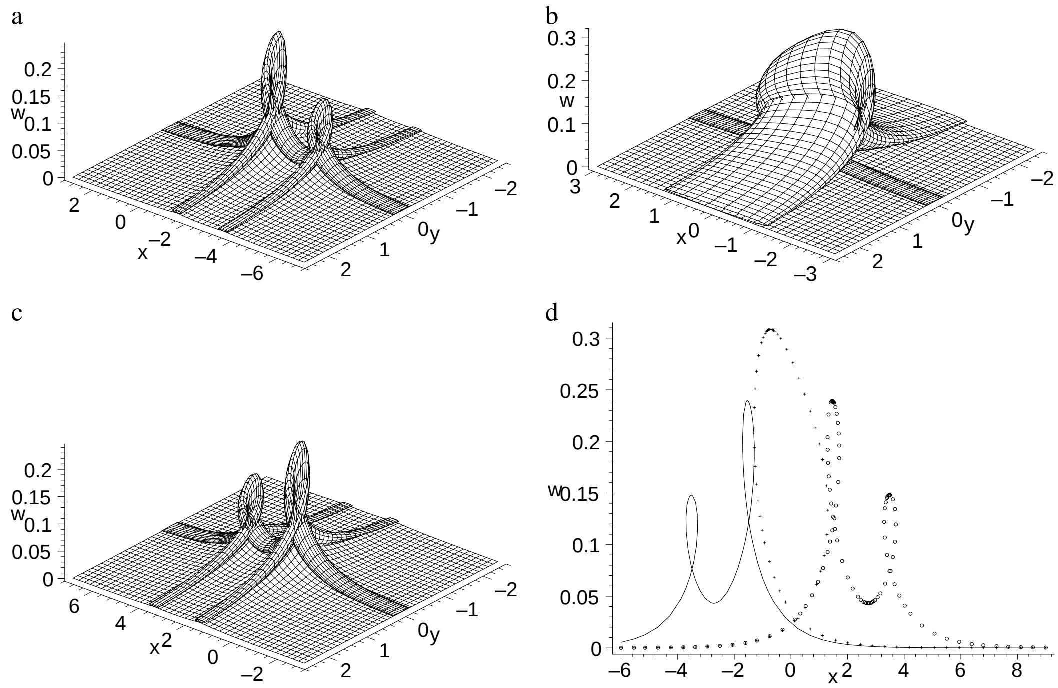

Fig. 3. Elastic interaction between special folded solitary waves w with conditions (38) at (a) $t = -5$; (b) $t = 0$; (c) $t = 5$. (d) Corresponding sectional view at $y = 0$; solid line, dots, circles denote before, in and after collision, respectively.

From Figs. 3a – c, we can see that the interaction between two-folded solitary waves is completely elastic. This is guaranteed by the completely elastic interaction property of the (1+1)-dimensional fields p and q. The conditions (25) in the case are satisfied:

$$\Omega_1^+ - \Omega_1^- = 0, \quad \Omega_2^+ - \Omega_2^- = 0. \tag{39}$$

One of the velocities of the two-folded solitary waves is fixed as zero, which makes it easy to determine their phase shifts. Clearly before the interaction, the static folded solitary wave is located at $x = -1.5$, while after the interaction, it shifts to $x = 1.5$. The total phase shift thus is

$$\Delta_1^+ - \Delta_1^- = G_2(-\infty) - G_2(+\infty) = 3. \tag{40}$$

6. Summary

In summary, by means of an extended tanh approach, the (2+1)-dimensional general Sasa-Satsuma system was successfully solved. Based on the derived variable separated solutions with two arbitrary, characteristic, lower-dimensional functions p and q, we have found rich localized excitations by selecting the arbitrary functions appropriately. Especially, some elastic and nonelastic interactions among the special folded solitary waves were investigated both analytically and graphically. The explicit phase shifts for all the local excitations offered by the common formula have been given and applied to these interactions in detail. The different choices of the arbitrary functions p and q in (27) and (28) corresponded to the different choices of the boundary conditions of those fields with nonzero boundary conditions. That means, in some sense, the dromions, folded solitary waves, and other types of localized excitations for some physical quantities are remote-controlled by some other quantities, see Figure 4. w is a dromion, while u is 3-solitoff with the same choices

$$\begin{aligned} p &= \frac{1 + 3\exp(x) + 3\exp(e_1 x)}{3 + e_2 \exp(x) + e_2 \exp(e_1 x)}, \\ q &= \exp(y) + \exp(e_1 y), \\ e_1 &= \frac{1}{3}, \quad e_2 = \frac{1}{2}. \end{aligned} \tag{41}$$

Furthermore, due to the arbitrariness of p and q, we can construct not only solitons but also chaos, although

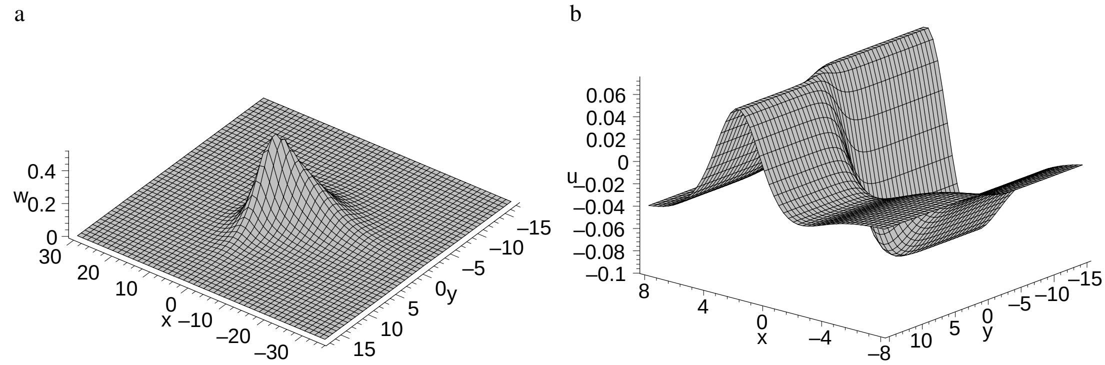

Fig. 4. (a) Dromion for the field; (b) 3-solitoff for the field w.

one usually argues that chaos is the basic behaviour of nonintegrable models. If one of p and q is chosen to be a localized function while the other one is a chaotic solution of some (1+1)-dimensional [or (0+1)-dimensional] nonintegrable model – for example, we set p to be the solution of the Lorentz system

$$p_{\zeta\zeta\zeta} = \frac{p_{\zeta\zeta} p_{\zeta}}{p} - (p^2 + b(c+1))p_{\zeta} - (b+c+1)p_{\zeta\zeta} + (b(a-1) - p^2)cp, \quad \zeta = x + \omega t, \tag{42}$$

where a, b, c are all arbitrary constants – in this case, the localized excitations are chaotic in time and space.

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